Let $f(x)=\dfrac{x^5}{7+x}$.

SCORE: _____ / 15 PTS

[a] Find a power series for $f(x)$. Write your answer is sigma notation.
NOTE: You do **NOT** need to use differentiation **NOR** the binomial series.

$$\frac{x^5}{7+x} = \frac{x^5}{7}\left(\frac{1}{1+\frac{x}{7}}\right) = \frac{x^5}{7}\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+5}}{7^{n+1}}$$

$$= \sum_{n=5}^{\infty}\frac{(-1)^{n+1} x^n}{7^{n-4}}$$

[b] Find the interval of convergence for the power series for $f(x)$.
NOTE: You do **NOT** need to find a limit.

$$\left|-\frac{x}{7}\right| < 1 \implies |x| < 7$$

$$-7 < x < 7$$

Use the binomial series to write the Maclaurin series for $f(x) = \sin^{-1} x$ in sigma notation. Simplify all coefficients as shown in lecture.

SCORE: _____ / 25 PTS

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} = (1+(-x^2))^{-\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}x^{2n}$$

$$\sin^{-1}x = C + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= C + \int \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}x^{2n}\,dx = C + \sum_{n=0}^{\infty}\int(-1)^n\binom{-\frac{1}{2}}{n}x^{2n}\,dx$$

$$= C + \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\frac{x^{2n+1}}{2n+1}$$

FOR $n > 0$,

$$\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)}{n!}$$

$$= \frac{(-1)^n(1\cdot3\cdot5\cdots(2n-1))}{2^n n!}$$

$$\binom{-\frac{1}{2}}{0} = 1$$

$$= C + x + \sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n\cdot n!\,(2n+1)}x^{2n+1}$$

$$\sin^{-1}0 = C$$

$$0 = C$$

$$\sin^{-1}x = x + \sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n\cdot n!\,(2n+1)}x^{2n+1}$$

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n-3)(x+7)^n}{n^2+4}$. **SCORE: _____ / 35 PTS**

$$\lim_{n\to\infty} \left| \frac{(2(n+1)-3)(x+7)^{n+1}}{(n+1)^2+4} \cdot \frac{n^2+4}{(2n-3)(x+7)^n} \right| = \lim_{n\to\infty} \left| \frac{2n-1}{2n-3} \cdot \frac{n^2+4}{n^2+2n+5} (x+7) \right|$$

$$= |x+7| < 1$$

IF $-1 < x+7 < 1$

$$-8 < x < -6$$

$x=-6$: $\sum_{n=0}^{\infty} \frac{2n-3}{n^2+4}$ COMPARE TO $\sum_{n=0}^{\infty} \frac{2n}{n^2} = \sum_{n=0}^{\infty} \frac{2}{n}$

$\frac{2n-3}{n^2+4} > 0$ IF $n > \frac{3}{2}$ AND $\frac{2}{n} > 0$ IF $n > 0$

$$\lim_{n\to\infty} \frac{\frac{2n-3}{n^2+4}}{\frac{2}{n}} = \lim_{n\to\infty} \frac{2n^2-3n}{2n^2+8} = 1 \neq 0$$

$\sum_{n=0}^{\infty} \frac{2}{n}$ DIVERGES $(p=1)$

SO $\sum_{n=0}^{\infty} \frac{2n-3}{n^2+4}$ DIVERGES BY LIMIT COMPARISON TEST

$x=-8$: $\sum_{n=0}^{\infty} \frac{(-1)^n(2n-3)}{n^2+4}$

$\frac{2n-3}{n^2+4} > 0$ IF $n > \frac{3}{2}$

$$\lim_{n\to\infty} \frac{2n-3}{n^2+4} = 0$$

$$\frac{d}{dx} \frac{2x-3}{x^2+4} = \frac{2(x^2+4)-2x(2x-3)}{(x^2+4)^2} = \frac{-2x^2+6x+8}{(x^2+4)^2} = \frac{-2(x-4)(x+1)}{(x^2+4)^2} < 0$$

IF $x > 4$

SO $\left\{\frac{2n-3}{n^2+4}\right\}$ IS DECREASING

SO $\sum_{n=0}^{\infty} \frac{(-1)^n(2n-3)}{n^2+4}$ CONVERGES BY ALTERNATING SERIES TEST

INTERVAL $= [-8, -6)$ OR $-8 \le x < -6$

Find the sum of the series $\frac{\pi}{5}+\frac{\pi^2}{2\times 25}+\frac{\pi^3}{3\times 125}+\frac{\pi^4}{4\times 625}+\cdots$. **SCORE: _____ / 10 PTS**

$$-\left[\left(-\frac{\pi}{5}\right)-\frac{\left(-\frac{\pi}{5}\right)^2}{2}+\frac{\left(-\frac{\pi}{5}\right)^3}{3}-\frac{\left(-\frac{\pi}{5}\right)^4}{4}+\cdots\right]$$

$$=-\ln\left(1+-\frac{\pi}{5}\right)$$

$$=-\ln\left(1-\frac{\pi}{5}\right)$$

Determine if the series $\sum_{n=2}^{\infty} \frac{(\ln n)(\sin n)}{n^2}$ converges. **SCORE: _____ / 30 PTS**

$$0 \leq \left| \frac{(\ln n)(\sin n)}{n^2} \right| \leq \frac{\ln n}{n^2}$$

$f(x) = \frac{\ln x}{x^2}$ IS POSITIVE + CONTINUOUS ON $[2, \infty)$

$f'(x) = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} < 0$ IF $x > \sqrt{e}$

SO f IS DECREASING ON $[2, \infty)$

u	dv
$\ln x$	x^{-2}
x^{-1}	$-x^{-1}$
1	$-x^{-2}$
0	x^{-1}

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{N\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right)\Big|_2^N$$

$$= \lim_{N\to\infty} \left(\frac{-\ln N}{N} - \frac{1}{N} + \frac{\ln 2}{2} - \frac{1}{2} \right)$$

$$= 0 - 0 + \frac{\ln 2}{2} - \frac{1}{2}$$

CONVERGES

$$\lim_{N\to\infty} \frac{-\ln N}{N} \quad \frac{-\infty}{\infty}$$

$$= \lim_{N\to\infty} \frac{-\frac{1}{N}}{1}$$

$$= 0$$

SO $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ CONVERGES BY INTEGRAL TEST

SO $\sum_{n=2}^{\infty} \left| \frac{(\ln n)(\sin n)}{n^2} \right|$ CONVERGES BY COMPARISON TEST

SO $\sum_{n=2}^{\infty} \frac{(\ln n)(\sin n)}{n^2}$ CONVERGES BY ABSOLUTE CONVERGENCE TEST

Find the first 4 **non-zero** terms of the Taylor series for $f(x) = \sec x$ about $x = \frac{\pi}{4}$. **SCORE: _____ / 20 PTS**

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\sec x$	$\sqrt{2}$
1	$\sec x \tan x$	$\sqrt{2}$
2	$\sec x \tan^2 x + \sec^3 x$	$\sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$
3	$\sec x \tan^3 x + 2\sec^3 x \tan x + 3\sec^3 x \tan x$ $= \sec x \tan^3 x + 5\sec^3 x \tan x$	$\sqrt{2} + 5 \cdot 2\sqrt{2} = 11\sqrt{2}$

$\sec\frac{\pi}{4} = \sqrt{2}$
$\tan\frac{\pi}{4} = 1$

$$\sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3$$

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!(x-1)^n}{3^{2n}(n!)^2}$. **SCORE: _____ / 15 PTS**

$$\lim_{n\to\infty} \left| \frac{(2(n+1))!(x-1)^{n+1}}{3^{2(n+1)}((n+1)!)^2} \cdot \frac{3^{2n}(n!)^2}{(2n)!(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)(x-1)}{3^2 (n+1)^2} \right|$$

$$= |x-1| \lim_{n\to\infty} \frac{1}{9} \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \frac{4}{9}|x-1| < 1$$

IF $|x-1| < \frac{9}{4}$

RADIUS $= \frac{9}{4}$